Soft Comput
DOI 10.1007/s00500-014-1314-9

METHODOLOGIES AND APPLICATION

A Runge–Kutta method with reduced number of function evaluations to solve hybrid fuzzy differential equations

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Abstract In this paper, we employ a numerical algorithm to solve first-order hybrid fuzzy differential equation (HFDE) based on the high order Runge–Kutta method. It is assumed that the user will evaluate both f and f' readily, instead of the evaluations of f only when solving the HFDE. We present a $O(h^4)$ method that requires only three evaluations of f. Moreover, we consider the characterization theorem of Bede to solve the HFDE numerically. The convergence of the method will be proven and numerical examples are shown with a comparison to the conventional solutions.

Keywords Fuzzy ordinary differential equation · Hybrid system · Bede's characterization theorem · High order Runge–Kutta method · Seikkala derivative

Communicated by V. Loia.

This research is supported by the Program Rakan Penyelidikan UM Grant CG065-2013, H-00000-61000-E13110 from the University of Malaya.

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1 Introduction

The study of fuzzy differential equations (FDEs) forms a suitable setting for mathematical modelling of the real world in various fields including synchronize hyperchaotic systems (Zhang et al. 2005), control chaotic systems (Feng and Chen 2005; Jiang et al. 2005), medicine (Abbod et al. 2001; Barro and Marn 2002), bioinformatics and computational biology (Casasnovas and Rossell 2005; Chang and Halgamuge 2002). A thorough theoretical research of fuzzy initial value problem was studied by Kaleva (1987), Seikkala (1987), Kloeden (1991) and Wu et al. (1996) which was then followed up by Salahshour et al. (2012a,b) using the fuzzy fractional differential equations. Moreover, applications of numerical and analytical methods such as the fuzzy Euler method (Ma et al. 1999), Adams-Bashforth, Adams-Moulton, predictor–corrector (Allahviranloo et al. 2007), power series (Allahviranloo et al. 2011), Runge–Kutta (Palligkinis et al. 2009), Laplace transform (Salahshour et al. 2012a; Salahshour and Allahviranloo 2013), fuzzy differential transform method (Salahshour and Allahviranloo 2013) and orthogonal polynomials (Ahmadian et al. 2013) in various kinds of FDEs were presented to extend the implementation of numerical and analytical methods for the fuzzy ordinary differential equations (FODE).

Particularly, in recent years the use of hybrid fuzzy differential equations (HFDEs) has increased drastically. One of the main factors is due to its natural way to model control systems with embedded uncertainty (containing fuzzy valued functions) that are capable of controlling complex systems which consist of discrete event dynamics as well as continuous time dynamics. For instance, Pederson and Sambandham (2007, 2008, 2009) investigated the numerical solution of HFDEs, using the Euler and Runge–Kutta methods. Similarly, Prakash and Kalaiselvi (2009); Kima

Published online: 14 June 2014

and Sakthivel (2012) studied the predictor–corrector method for HFDEs, and Allahviranloo and Salahshour (2011) investigated the numerical solution of HFDEs, using the Euler method under characterization theorem and Bede's differentiability.

Bede (2008) proved a characterization theorem which states that under certain conditions a FDE under the Hukuhara differentiability is equivalent to a system of ordinary differential equations (ODEs). Moreover, Bede also noticed that this characterization theorem can aid to solve FDEs numerically through converting the FDEs to a system of ODEs, which later could be solved by numerical methods. In this paper, using the characterization theorem, we generalize a fourth-order Runge–Kutta method that originally presented to solve the HFDEs. That is, we substitute the original initial value problem with two parametric hybrid ordinary differential systems. Then, the extension of Bede's characterization theorem for HFEDs, which was investigated by Pederson and Sambandham (2009), is employed to generalize the derivatives. Finally, these results are applied to solve the HFDEs numerically by the fourth-order reduced Runge–Kutta (RRK) method.

The rest of the paper is structured as follows: We revisit the preliminary in Sect. 2; the fourth-order RRK method will be explained in Sect. 3. We study HFDEs using the concept of characterization theorem in Sect. 3.1. In Sect. 4, the fourth-order fuzzy RRK method be proposed. Numerical experiments are provided in Sect. 4.1 and compared with other methods. This is followed by a complete error analysis. At the end of the paper, we present some conclusions and our future work.

2 Preliminaries

We give some definitions and introduce the necessary notation in this section which will be used throughout the paper. For detail, readers are encouraged to refer to Xu et al. (2007). We consider $\mathbb{R}$ as the set of all real numbers, and a fuzzy number is a mapping $u : \mathbb{R} \to [0, 1]$ with the following properties:

(a) u is upper semi-continuous,
(b) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0, 1]$,
(c) u is normal, i.e., $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,
(d) $\text{supp}\ u = \{x \in \mathbb{R} | u(x) > 0\}$ is the support of the u, and its closure cl(supp u) is compact.

Let $\mathbb{E}$ be the set of all fuzzy numbers on $\mathbb{R}$. The r-level set of a fuzzy number $u \in \mathbb{E}$, $0 \leq r \leq 1$, denoted by $[u]_r$, is defined as

$$[u]_r = \begin{cases} \{x \in \mathbb{R} | u(x) \geq r\} & \text{if } 0 < r \leq 1 \\ cl(\text{supp}\ u) & \text{if } r = 0 \end{cases}$$

It is clear that the r-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(r), \overline{u}(r)]$, where $\underline{u}(r)$ denotes the left-hand endpoint of $[u]_r$ and $\overline{u}(r)$ denotes the right-hand endpoint of $[u]_r$. Since each $y \in \mathbb{R}$ can be regarded as a fuzzy number $\widetilde{y}$ is defined:

$$\tilde{y}(t) = \begin{cases} 1 & \text{if} \quad t = y \\ 0 & \text{if} \quad t \neq y \end{cases}$$

For $u, v \in \mathbb{E}$ and $\lambda \in \mathbb{R}$, the sum $(u + v)$ and the product $(\lambda \odot u)$ are defined by $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$; $[\lambda \odot u]^\alpha = \lambda[u]^\alpha, \forall \alpha \in [0, 1]$, where $[u]^\alpha + [v]^\alpha$ represents the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[u]^\alpha$ is the usual product between a scalar and a subset of $\mathbb{R}$.

The Hausdorff distance fuzzy numbers are given by $D : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}_+ \bigcup 0$,

$$D(u, v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)|\},$$

It is easy to see that D is a metric in $\mathbb{E}$ and has the following properties (Dubios and Prade 1982):

(i) $D(u \oplus w, v \oplus w) = D(u, v), \quad \forall u, v, w \in \mathbb{E},$,
(ii) $D(k \odot u, k \odot v) = |k| D(u, v), \quad \forall k \in \mathbb{R}, u, v \in \mathbb{E},$,
(iii) $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e), \quad \forall u, v, w \in \mathbb{E},$,
(iv) $(\mathbb{E}, D)$ is a complete metric space.

Definition 1 Let $f : \mathbb{R} \to \mathbb{E}$ be a fuzzy valued function. If for arbitrary fixed $t_0 \in \mathbb{R}$ and for any given $\epsilon > 0$, there exists $\delta > 0$ such that

$$D(f(t), f(t_0)) < \varepsilon,$$

as $D(t, t_0) < \delta$, then $f(t)$ is called continuous in t_0 (Guang-Quan 1991).

Initially, the H-derivative (Hukuhara differentiability) for fuzzy mappings was introduced by Puri and Ralescu (1983) which is based on the H-difference sets, as follows:

Definition 2 Let $x, y \in \mathbb{E}$. If there exists $z \in \mathbb{E}$ such that $x = y \oplus z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$.

In this paper, the sign "$\ominus$" stands for H-difference, and also note that $x \ominus y \neq x + (-1)y$.

Definition 3 Let $f : \mathbb{R} \to \mathbb{E}$ be a fuzzy function. We say that f is differentiable at $t_0 \in \mathbb{R}$, if there exists an element $f^{'}(t_0) \in \mathbb{E}$ such that limits

$$\lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h}$$

exist and are equal to $f^{'}(t_0)$. Here, the limits are taken in the metric space $(\mathbb{E}, D)$, since we have defined $h^{-1} \odot (f(t_0) \ominus f(t_0-h))$ and $h^{-1} \odot (f(t_0+h) \ominus f(t_0))$. Next, we present the Bede (2008) characterization theorem (note that, $\|.\|$ denotes the usual Euclidean norm).

Theorem 1 (Characterization Theorem) *Let us consider the fuzzy initial value problem (FIVP)*

$$\begin{cases} x' = f(t, x), \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $f : [t_0, t_0 + a] \times \mathbb{E} \to \mathbb{E}$ is such that

(i) $[f(t, x)]^r = [\underline{f}^r(t, \underline{x}, \overline{x}), \overline{f}^r(t, \underline{x}, \overline{x})]$,
(ii) $\underline{f}^r$ and $\overline{f}^r$ are equicontinuous (that is, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|\underline{f}^r(t, x, y) - \underline{f}^r(t_1, x_1, y_1)| < \varepsilon$ and $|\overline{f}^r(t, x, y) - \overline{f}^r(t_1, x_1, y_1)| < \varepsilon$ for all $r \in [0, 1]$, whenever $(t, x, y), (t_1, x_1, y_1) \in [t_0, t_0 + a] \times \mathbb{R}^2$ and $\|(t, x, y) - (t_1, x_1, y_1)\| < \delta$) and uniformly bounded on any bounded set,
(iii) there exists an $L > 0$ such that
$|\underline{f}^r(t, x_1, y_1) - \underline{f}^r(t, x_2, y_2)| \leq L \max\{|x_2 - x_1|, |y_2 - y_1|\}$ for all $r \in [0, 1]$,
$|\overline{f}^r(t, x_1, y_1) - \overline{f}^r(t, x_2, y_2)| \leq L \max\{|x_2 - x_1|, |y_2 - y_1|\}$ for all $r \in [0, 1]$.

Then, the FIVP (1) and system of ODEs

$$\begin{cases} (\underline{x}^r(t))' = \underline{f}^r(t, \underline{x}^r, \overline{x}^r) \\ (\overline{x}^r(t))' = \overline{\overline{f}}^r(t, \underline{x}^r, \overline{x}^r) \\ \underline{x}^r(t_0) = (\underline{x_0^r}) \\ \overline{x}(t_0) = (\overline{\overline{x_0^r}}) \end{cases} \tag{2}$$

are equivalent.

3 A fourth-order Runge–Kutta method with three functional evaluations

Consider the initial value problem

$$\begin{aligned} y' &= f(x, y), \\ y(x_0) &= y_0 \quad \text{with } (x_0, y_0) \in D \end{aligned} \tag{3}$$

at which we assume that $f(x, y)$ has derivatives to the fourth order in domain D in $\mathbb{R}^{n+1}$ where $x \in \mathbb{R}, y \in \mathbb{R}^n$ and $(x, y) \in D$. Also we consider that $||f(x, y_1) - f(x, y_2)||_2 \leq L||y_1 - y_2||_2$, thus the problem (3) has a unique local solution.

Much efforts have been made to improve the order of Runge–Kutta methods by means of increasing the number of terms in the Taylor series expansion. This increases the number of function evaluations accordingly. The direct use of the Jacobian matrix in an integrator for stiff problems has been proposed by some authors (Jackiewicz and Tracogna 1995; Enright 1974; Rosenbrock 1963). Goeken and Johnson (2000) introduced new terms involving higher order derivatives of f in the Runge–Kutta k_i terms ($i > 1$) to achieve a higher order of accuracy without a corresponding increase in the evaluations of f, but with the addition of evaluations or approximations of f' for third, fourth and fifth order method. The advantage of this method is that it has lower functional evaluation which improved the effectiveness of the method in comparison to the classical Runge–Kutta. Nonetheless, it can be applied in both the autonomous and non-autonomous systems.

Considering the problem in autonomous form, the fourth-order formula has the following form:

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2 + b_3 k_3, \tag{4}$$

where

$$\begin{aligned} k_1 &= hf(y_n), \\ k_2 &= hf(y_n + a_{21}k_1 + ha_{22}f_y(y_n)k_1), \\ k_3 &= hf(y_n + a_{31}k_1 + a_{32}k_2 + ha_{33}f_y(y_n)k_1 \\ &\quad + ha_{34}f_y(y_n)k_2). \end{aligned} \tag{5}$$

Specific nonzero constants, in the fourth-order RRK method for autonomous systems, are

$$b_1 = \frac{1}{6},\ b_2 = \frac{2}{3},\ b_3 = \frac{1}{6},\ a_{21} = \frac{1}{2},\ a_{22} = \frac{1}{8},\ a_{31} = -1,\ a_{32} = 2,\ a_{33} = -\frac{1}{2}.$$

The specific formula of interest is

$$y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{2}{3}k_2 + \frac{1}{6}k_3. \tag{6}$$

Remark 1 The proposed new formulae with f' such as (4) are more efficient for cases where f' is not computationally expensive to evaluate than f. For example, with a linear system of equations $y' = Ay$, $f' = Ay'$ requires similar amount of time to compute as does $f = Ay$.

3.1 Non-autonomous derivations

If we proceed as above for $y' = f(x, y)$, we need to augment the terms involving $f_y k_i$ with hf_x. We can vary the terms used to match parameters. For example, if the equation is scalar, it is possible to use an f_y term in the x displacement rather than a f_x term in the y displacement. Since $f' = f_y f + f_x$ and $f_y f$ is needed in the y displacement anyway, the exact computation of f_y is easier than the computation of f'. The following method uses f and f_y:

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2 + b_3 k_3, \tag{7}$$

where

$$\begin{aligned}
k_1 &= hf(x_n, y_n),\\
k_2 &= hf(x_n + hc_{21} + h^2 c_{22} f_y, y_n + a_{21}k_1 + ha_{22} f_y k_1),\\
k_3 &= hf(x_n + hc_{31} + h^2 c_{32} f_y, y_n + a_{31}k_1 + a_{32}k_2\\
&\quad + ha_{33} f_y(y_n)k_1 + ha_{34} f_y(y_n)k_2),
\end{aligned} \tag{8}$$

where f_y is evaluated at (x_n, y_n). The nonzero constant coefficients, in the fourth-order RRK method for non-autonomous systems, are

$$\begin{aligned}
& b_1 = \frac{1}{6},\ b_2 = \frac{2}{3},\ b_3 = \frac{1}{6},\ c_{21} = a_{21} = \frac{1}{2},\\
& \quad c_{22} = a_{22} = \frac{3}{32},\\
& a_{31} = -\frac{1}{2},\ a_{32} = \frac{3}{2},\ a_{33} = -\frac{11}{32},\\
& \quad a_{34} = \frac{7}{32},\ c_{31} = 1,\ c_{32} = -\frac{1}{8}.
\end{aligned} \tag{9}$$

Remark 2 This method has the same stability equation as the classical fourth-order Runge–Kutta method. In fact, when using a functional evaluation of f' or f_y, new methods presented here have the same stability as the classical Runge–Kutta method of fourth order Goeken and Johnson (2000).

4 Hybrid fuzzy differential equation

In this paper, we consider the HFDE

$$\begin{aligned}
& x'(t) = f(t, x(t), \lambda_k(x_k)), \quad t \in [t_k, t_{k+1}],\ k = 0, 1, 2, \ldots\\
& x(t_0) = x_0,
\end{aligned} \tag{10}$$

where $\{t_k\}_{k=0}^{\infty}$ is strictly increasing and unbounded, x_k denotes $x(t_k)$, $f : [t_0, \infty) \times \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{E}$ is continuous and each $\lambda_k : \mathbb{E} \rightarrow \mathbb{E}$ is continuous. A solution to (10) will be a function $x : [t_0, \infty) \rightarrow \mathbb{E}$ satisfying (10). For $k = 0, 1, 2, \ldots$, let $f_k : [t_k, t_{k+1}] \times \mathbb{E} \rightarrow \mathbb{E}$, where $f_k(t, x_k(t)) = f(t, x_k(t), \lambda_k(x_k))$. The HFDE (10) can be written as

$$x'(t) = \begin{cases}
x_0'(t) = f(t, x_0(t), \lambda_0(x_0)) = f_0(t, x_0(t)),\\
\quad t_0 \le t \le t_1,\\
x_1'(t) = f(t, x_1(t), \lambda_1(x_1)) = f_1(t, x_1(t)),\\
\quad t_1 \le t \le t_2,\\
\vdots\\
x_k'(t) = f(t, x_k(t), \lambda_k(x_k)) = f_k(t, x_k(t)),\\
\quad t_k \le t \le t_{k+1},\\
\vdots
\end{cases} \tag{11}$$

and a solution of (10) can be expressed as

$$x(t) = \begin{cases}
x_0(t), & t_0 \le t \le t_1,\\
x_1(t), & t_1 \le t \le t_2,\\
\vdots\\
x_k(t), & t_k \le t \le t_{k+1},\\
\vdots
\end{cases} \tag{12}$$

Using the Bede (2008) characterization theorem, generalized the following characterization theorem for HFDE IVPs:

Theorem 2 *Consider the HFDE IVP (10) expanded as (11) where for $k = 0, 1, 2, \ldots$, each $f_k : [t_k, t_{k+1}] \times \mathbb{E} \rightarrow \mathbb{E}$, is such that:*

(i) $[f_k(t, x)]^r = [(\underline{f}_k)^r(t, \underline{x}^r, \overline{x}^r), (\overline{f}_k)^r(t, \underline{x}^r, \overline{x}^r)]$,

(ii) *$(\underline{f}_k)^r$ and $(\overline{f}_k)^r$ are equicontinuous (that is, for any $\varepsilon > 0$ there is a $\delta_k > 0$ such that $|\underline{f}_k^r(t, x, y) - \underline{f}_k^r(t_1, x_1, y_1)| < \varepsilon$ and $|\overline{f}^r(t, x, y) - \overline{f}^r(t_1, x_1, y_1)| < \varepsilon$ for all $r \in [0, 1]$, whenever $(t, x, y), (t_1, x_1, y_1) \in [t_k, t_{k+1}] \times \mathbb{R}^2$ and $\| (t, x, y) - (t_1, x_1, y_1) \| < \delta_k(\epsilon)$) and uniformly bounded on any bounded set,*

(iii) *there exists an $L_k > 0$ such that*
$|\underline{f}_k^r(t, x_1, y_1) - \underline{f}_k^r(t, x_2, y_2)| \le L_k \max\{|x_2 - x_1|, |y_2 - y_1|\}$ for all $r \in [0, 1]$,
$|\overline{f}_k^r(t, x_1, y_1) - \overline{f}_k^r(t, x_2, y_2)| \le L_k \max\{|x_2 - x_1|, |y_2 - y_1|\}$ for all $r \in [0, 1]$.

Then, (10) and the hybrid system of ODEs

$$\begin{cases}
((\underline{x}_k)^r(t))' = \underline{f}_k^r(t, (\underline{x}_k)^r(t), (\overline{x}_k)^r(t))\\
((\overline{x}_k)^r(t))' = \overline{f}_k^r(t, (\underline{x}_k)^r(t), (\overline{x}_k)^r(t))\\
(\underline{x}_k)^r(t_k) = (\underline{x}_{k-1})^r(t_k), \ \text{if } k > 0, (\underline{x}_0)^r(t_0) = (\underline{x}_0)^r,\\
(\overline{x}_k)^r(t_k) = (\overline{x}_{k-1})^r(t_k), \ \text{if } k > 0, (\overline{x}_0)^r(t_0) = (\overline{x}_0)^r,
\end{cases} \tag{13}$$

are equivalent.

Proof See Pederson and Sambandham (2009). □

4.1 Fourth-order fuzzy Reduced Runge–Kutta method

In this section, we will present the RRK method mentioned in Sect. 3 for solving the FHDE. The RRK methods algorithm exploit the use of higher order derivatives, specifically f'. To numerically solve the hybrid system of ordinary differential system in $[t_0, t_1], [t_1, t_2], \ldots, [t_k, t_{k+1}], \ldots$, for $\alpha \in [0, 1]$, we will replace each interval $[t_k, t_{k+1}]$ by a set of $N_k + 1$ regularly spaced grid points. The grid points on $[t_k, t_{k+1}]$ will be $t_{k,n} = t_k + nh_k$ where $h_k = \frac{t_{k+1} - t_k}{N_k}$ and $0 \le n \le N_k$, at which the exact solution $x(t; r) = (\underline{x}(t; r), \overline{x}(t; r))$ is approximated by some $(\underline{y}_k(t; r), \overline{y}_k(t; r))$ and $(\underline{Y}_k(t; r), \overline{Y}_k(t; r)) \equiv (\underline{x}(t; r), \overline{x}(t; r))$. $(\underline{Y}_k(t; r), \overline{Y}_k(t; r))$ and $(\underline{y}_k(t; r), \overline{y}_k(t; r))$

may be denoted respectively by $(\underline{Y}_{k,n}\ (t;r), \overline{Y}_{k,n}(t;r))$ and $(\underline{y}_{k,n}(t;r), \overline{y}_{k,n}(t;r))$.

On the other hand, Theorem 1 states that a FDE is equivalent to a system of ordinary differential equations under certain conditions. By the same reasoning, we may use Theorem 2 to solve hybrid fuzzy initial value problem (HFIVP) numerically by applying the fourth-order RRK method.

We will apply Theorems 1 and 2 for (13). Firstly, we should convert (10) to (13) and after that using the method developed in Sect. 3, the hybrid fuzzy equations' system (13) is solved. The generalized RRK method based on the fourth-order approximation of $\underline{Y}'_k(t;r)$, $\overline{Y}'_k(t;r)$ and Eqs. (7), (8) and (13) are attained as follows:

$$\begin{cases} \underline{y}_{k,n+1}(r) = \underline{y}_{k,n}(r) + F_k(t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)), \\ \overline{y}_{k,n+1}(r) = \overline{y}_{k,n}(r) + G_k(t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)), \\ (\underline{y}_{0,0})^r = (\underline{x}_0)^r, \quad (\underline{y}_{k,0})^r = (\underline{y}_{k-1,N_{k-1}})^r, \\ (\overline{y}_{0,0})^r = (\overline{x}_0)^r, \quad (\overline{y}_{k,0})^r = (\overline{y}_{k-1,N_{k-1}})^r. \end{cases} \tag{14}$$

which we define

$$\begin{cases} F_k(t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)) = \frac{1}{6}\underline{k}_1(t_{k,n}, y_{k,n}(r)) \\ \quad +\frac{2}{3}\underline{k}_2(t_{k,n}, y_{k,n}(r)) + \frac{1}{6}\underline{k}_1(t_{k,n}, y_{k,n}(r)), \\ G_k(t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)) = \frac{1}{6}\overline{k}_1(t_{k,n}, y_{k,n}(r)) \\ \quad +\frac{2}{3}\overline{k}_2(t_{k,n}, y_{k,n}(r)) + \frac{1}{6}\overline{k}_1(t_{k,n}, y_{k,n}(r)), \end{cases} \tag{15}$$

at which

$$\begin{cases} \underline{k}_1(t_{k,n}, y_{k,n}(r)) = h_k \underline{f}(t_{k,n}, y_{k,n}(r), \lambda_k(y_k)), \\ \underline{k}_2(t_{k,n}, y_{k,n}(r)) = h_k \underline{f}(\underline{z}_{1_{k,n}}, \underline{z}_{2_{k,n}}), \\ \underline{k}_3(t_{k,n}, y_{k,n}(r)) = h_k \underline{f}(\underline{z}_{3_{k,n}}, \underline{z}_{4_{k,n}}), \end{cases}$$

such that

$$\begin{cases} \underline{z}_{1_{k,n}} = t_{k,n} + \frac{1}{2}h_k + \frac{3}{32}h_k^2 \underline{f}_y(t_{k,n}, y_{k,n}(r)), \\ \underline{z}_{2_{k,n}} = y_{k,n}(r) + \frac{1}{2}\underline{k}_1(t_{k,n}, y_{k,n}(r)) \\ \quad +\frac{3}{32}h_k \underline{f}_y(t_{k,n}, y_{k,n}(r))\underline{k}_1(t_{k,n}, y_{k,n}(r)), \\ \underline{z}_{3_{k,n}} = t_{k,n} + h_k - \frac{1}{8}h_k^2 \underline{f}_y(t_{k,n}, y_{k,n}(r)), \\ \underline{z}_{4_{k,n}} = y_{k,n}(r) + -\frac{1}{2}\underline{k}_1(t_{k,n}, y_{k,n}(r)) \\ \quad +\frac{3}{2}\underline{k}_2(t_{k,n}, y_{k,n}(r)) + \\ \quad -\frac{11}{32}h_k \underline{f}_y(t_{k,n}, y_{k,n}(r))\underline{k}_1(t_{k,n}, y_{k,n}(r)) \\ \quad +\frac{7}{32}h_k \underline{f}_y(t_{k,n}, y_{k,n}(r))\underline{k}_2(t_{k,n}, y_{k,n}(r)). \end{cases}$$

Also, we have

$$\begin{cases} \overline{k}_1(t_{k,n}, y_{k,n}(r)) = h_k \overline{f}(t_{k,n}, y_{k,n}(r), \lambda_k(y_k)), \\ \overline{k}_2(t_{k,n}, y_{k,n}(r)) = h_k \overline{f}(\overline{z}_{1_{k,n}}, \overline{z}_{2_{k,n}}), \\ \overline{k_3}(t_{k,n}, y_{k,n}(r)) = h_k \overline{f}(\overline{z}_{3_{k,n}}, \overline{z}_{4_{k,n}}), \end{cases}$$

where

$$\begin{cases} \overline{z}_{1_{k,n}} = t_{k,n} + \frac{1}{2}h_k + \frac{3}{32}h_k^2 \overline{f}_y(t_{k,n}, y_{k,n}(r)), \\ \overline{z}_{2_{k,n}} = y_{k,n}(r) + \frac{1}{2}\overline{k}_1(t_{k,n}, y_{k,n}(r)) \\ \quad +\frac{3}{32}h_k \overline{f}_y(t_{k,n}, y_{k,n}(r))\overline{k}_1(t_{k,n}, y_{k,n}(r)), \\ \overline{z}_{3_{k,n}} = t_{k,n} + h_k - \frac{1}{8}h_k^2 \overline{f}_y(t_{k,n}, y_{k,n}(r)), \\ \overline{z}_{4_{k,n}} = y_{k,n}(r) + -\frac{1}{2}\overline{k}_1(t_{k,n}, y_{k,n}(r)) + \frac{3}{2}\overline{k}_2(t_{k,n}, y_{k,n} \\ \quad (r)) + -\frac{11}{32}h_k \overline{f}_y(t_{k,n}, y_{k,n}(r))\overline{k}_1(t_{k,n}, y_{k,n}(r)) \\ \quad +\frac{7}{32}h_k \overline{f}_y(t_{k,n}, y_{k,n}(r))\overline{k}_2(t_{k,n}, y_{k,n}(r)). \end{cases}$$

We can get the exact solution of (13) as follows:

$$\begin{cases} Y_{k,n+1}(r) \approx Y_{k,n}(r) + F_k(t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)), \\ Y_{k,n+1}(r) \approx Y_{k,n}(r) + G_k(t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)). \end{cases}$$

The following results can be applied to show the convergence of these approximates (convergence is pointwise in r for a fixed k)

$$\lim_{h_0,\ldots,h_k\to 0} \underline{y}_{k,N_k}(r) = x(t_{k+1};r),$$
$$\lim_{h_0,\ldots,h_k\to 0} \overline{y}_{k,N_k}(r) = x(t_{k+1};r).$$

Lemma 1 *Let the sequence of numbers $\{W_n\}_{n=0}^N$ satisfy*

$$|W_{n+1}| \le A|W_n| + B, \quad 0 \le n \le N-1.$$

for some given positive constants A and B. Then

$$|W_n| \le A^n|W_0| + B\frac{A^n - 1}{A-1}, \quad 0 \le n \le N.$$

Lemma 2 *Let the sequence of numbers $\{W_n\}_{n=0}^N, \{V_n\}_{n=0}^N$ satisfy*

$$|W_{n+1}| \le |W_n| + A\ \max\{|W_n|, |V_n|\} + B,$$
$$|V_{n+1}| \le |V_n| + A\ \max\{|W_n|, |V_n|\} + B,$$

for some given positive constants A and B, denote

$$U_n = |W_n| + |V_n|, \quad 0 \le n \le N.$$

Then

$$|U_n| \le \overline{A}^n|U_0| + \overline{B}\frac{\overline{A}^n - 1}{\overline{A} - 1}, \quad 0 \le n \le N,$$

where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$.

Remark 3 Let $F_k(t_k, u, v)$ and $G_k(t_k, u, v)$ be the functions F_k and G_k of Eq. (14), where u and v are constants and $u \le v$. The domain where F_k and G_k are defined is, therefore,

$$K = \{(u, v) | -\infty < v < \infty, -\infty < u \le v\}, 0 \le t_k \le T.$$

Theorem 3 *Let $F_k(t_k, u, v)$ and $G_k(t_k, u, v)$ belong to $C^4(K)$ and let the partial derivatives of F_k and G_k be bounded over K. Also consider the systems (13) and (14), For a fixed $k \in \mathbb{Z}^+$ and $r \in [0, 1]$,*

$$\lim_{h_0,\ldots,h_k\to 0} \underline{y}_{k,N_k}(r) = x(t_{k+1};r),$$
$$\lim_{h_0,\ldots,h_k\to 0} \overline{y}_{k,N_k}(r) = x(t_{k+1};r). \tag{16}$$

Proof It is sufficient to show that:

$$\lim_{h_k\to 0} \underline{y}_{k,n}(r) = \underline{Y}_{k,n}(r),$$
$$\lim_{h_k\to 0} \overline{y}_{k,n}(r) = \overline{Y}_{k,n}(r). \tag{17}$$

where $t_{k,N_k} = T$. For $n = 0, \ldots, N_k - 1$, using exact value the following results will be obtained:

$$\underline{Y}_{k,n+1}(r) = \underline{Y}_{k,n}(r) + F_k(t_{k,n}, \underline{Y}_{k,n}(r), \overline{Y}_{k,n}(r)) + \frac{7}{256} h_k^5 MN^4 + O(h_k^6),$$
$$\overline{Y}_{k,n+1}(r) = \overline{Y}_{k,n}(r) + G_k(t_{k,n}, \underline{Y}_{k,n}(r), \overline{Y}_{k,n}(r)) + \frac{7}{256} h_k^5 MN^4 + O(h_k^6), \tag{18}$$

where Max $|f(t_{k,n}, Y_{k,n}(r))| < M$ and Max $|f_y(t_{k,n}, Y_{k,n}(r))| < N$. Denote

$$W_n = \underline{Y}_{k,n}(r) - \underline{y}_{k,n}(r), \quad V_n = \overline{Y}_{k,n}(r) - \overline{y}_{k,n}(r).$$

Hence, from (14) and (18), we obtain

$$W_{n+1} = W_n + F_k(\underline{Y}_{k,n}(r), \overline{Y}_{k,n}(r)) - F_k(\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)) + \frac{7}{256} h_k^5 MN^4 + O(h_k^6),$$
$$V_{n+1} = V_n + G_k(\underline{Y}_{k,n}(r), \overline{Y}_{k,n}(r)) - G_k(\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)) + \frac{7}{256} h_k^5 MN^4 + O(h_k^6).$$

Then, we have

$$|W_{n+1}| \le |W_n| + 2L_k h_k \max\{|W_n|, |V_n|\} + \frac{7}{256} h_k^5 MN^4 + O(h_k^6),$$
$$|V_{n+1}| \le |V_n| + 2L_k h_k \max\{|V_n|, |V_n|\} + \frac{7}{256} h_k^5 MN^4 + O(h_k^6),$$

for $t_k \in [0, T]$ and $L_k > 0$ is a bound for the partial derivatives of F_k and G_k. Thus, using Lemma 2

$$|W_{N_k}| \le (1 + 4L_k h_k)^n |U_0| + \left(\frac{7}{256} h_k^5 MN^4 + O(h_k^6)\right) \frac{(1 + 4L_k h_k)^n - 1}{4L_k h_k},$$
$$|V_{N_k}| \le (1 + 4L_k h_k)^n |U_0| + \left(\frac{7}{256} h_k^5 MN^4 + O(h_k^6)\right) \frac{(1 + 4L_k h_k)^n - 1}{4L_k h_k},$$

are the results, where $|U_0| = |W_0| + |V_0|$. Specifically,

$$|W_{N_k}| \le (1 + 4L_k h_k)^{N_k} |U_0| + \left(\frac{7}{256} h_k^4 MN^4 + O(h_k^6)\right) \frac{(1 + 4L_k h_k)^{T/h_k} - 1}{4L_k},$$
$$|V_{N_k}| \le (1 + 4L_k h_k)^{N_k} |U_0| + \left(\frac{7}{256} h_k^4 MN^4 + O(h_k^6)\right) \frac{(1 + 4L_k h_k)^{T/h_k} - 1}{4L_k},$$

is obtained. Since $W_0 = 0$, $V_0 = 0$, then we have

$$|W_{N_k}| \le \left(\tfrac{7}{256} MN^4 + O(h_k^6)\right) \tfrac{e^{4L_k T} - 1}{L_k} h_k^4 + O(h_k^6),$$
$$|V_{N_k}| \le \left(\tfrac{7}{256} MN^4 + O(h_k^6)\right) \tfrac{e^{4L_k T} - 1}{L_k} h_k^4 + O(h_k^6),$$

and if $h_k \to 0$ we get $W_{N_k} \to 0$, $V_{N_k} \to 0$ which completes the proof. □

5 Numerical results

Numerically, Pederson and Sambandham (2009, 2008) solved the examples below using the Runge–Kutta method. To verify that the new method is of the order claimed, these examples are solved using the fourth-order fuzzy RRK.

Remark 4 In remainder of the paper, the modulo (sometimes called modulus) operation finds the remainder of division of one number by another. Given two positive numbers, a (the dividend) and n (the divisor), a modulo n (abbreviated as $a$mod$n$) is the remainder of the Euclidean division of a by n.

Example 1 Consider the following hybrid fuzzy initial value problem

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), & t \in [t_k, t_{k+1}], \\ t_k = k, \quad k = 0, 1, 2, \ldots, \\ [x(0)]^r = [0.75 + 0.25r,\ 1.125 - 0.125r], & 0 \le r \le 1, \end{cases} \tag{19}$$

where

$$m(t) = \begin{cases} 2(t(\text{mod}1)) & \text{if } t(\text{mod}1) \le 0.5, \\ 2(1 - t(\text{mod}1)) & \text{if } t(\text{mod}1) \le 0.5, \end{cases}$$

in which

$$\lambda_k(\mu) = \begin{cases} \hat{0} & \text{if } k = 0, \\ \mu & \text{if } k \in \{1, 2, ...\}, \end{cases}$$

$$\hat{0}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \ne 0. \end{cases}$$

Next using the algorithm given in Sect. 4, we will solve the hybrid fuzzy equations corresponding to (19) by presenting the fourth-order Runge–Kutta to obtain numerical solutions to (19).

Case I: When $k = 0$, the solution of (19) in the interval $[0, 1]$:

When $k = 0$, the hybrid fuzzy initial value problem (4) becomes

$$\begin{cases} x'(t) = x(t), & t \in [0, 1], \\ [x(0)]^r = [0.75 + 0.25r, 1.125 - 0.125r]. \end{cases} \tag{20}$$

This is equivalent to the systems of ODEs

$$\begin{cases} \underline{x}'(t; r) = -\underline{x}(t; r), & t \in [0, 1], \\ \underline{x}(0; r) = 0.75 + 0.25r, & 0 \le r \le 1 \end{cases}$$
$$\times \begin{cases} \overline{x}'(t; r) = \overline{x}(t; r) & t \in [0, 1], \\ \overline{x}(0; r) = 1.125 - 0.125r, & 0 \le r \le 1 \end{cases}$$

Using Kaleva (1987) for $t \in [0, 1]$, the exact solution of (20) is given by

$$[x(t; r)] = [(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t],\quad 0 \le r \le 1.$$

The exact solution of (20) at $t = 1$ can be written as:

$$[x(1; r)] = [(0.75 + 0.25r)e, (1.125 - 0.125r)e],\quad 0 \le r \le 1.$$

By applying the fourth-order Runge–Kutta presented in Sect. 4 for autonomous system with $N = 2$ as to Ma et al. (1999); Palligkinis et al. (2009) and Theorem 1, (20) gives

$$y(1.0; r) = [(0.75 + 0.25r){c_1}^2, (1.125 - 0.125r){c_1}^2],\quad 0 \le r \le 1, \tag{21}$$

where

$$c_1 = \left(1 + \frac{1}{12} + \frac{41}{96} + \frac{45}{384}\right),$$

and $y(1.0; r)$ denotes an approximate solution of (20) at $t = 1$.

Case II: When $k = 1$, the solution of (20) in the interval $[1, 2]$:

The fuzzy problem (20) can be written as:

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_1(x(t_1)), & t \in [1, 2], \\ [x(0)]^r = [0.75 + 0.25r, 1.1250.125r]. \end{cases} \tag{22}$$

This is equivalent to the system of fuzzy differential equations

$$\begin{cases} \underline{x}'(t; r) = \underline{x}(t; r) + m(t)\underline{x}(1; r), \\ \underline{x}(1; r) = (0.75 + 0.25r)c_1^2, \end{cases} \tag{23a}$$

$$\begin{cases} \overline{x}'(t; r) = \overline{x}(t; r) + m(t)\overline{x}(1; r), \\ \overline{x}(1; r) = (1.125 - 0.125r)c_1^2. \end{cases} \tag{23b}$$

Then, using the method developed in Sect. 4, we solve the hybrid ODE systems (23) corresponding to (22), numerically.

By the fourth-order Runge–Kutta method, (14) and (15) for ODEs with $N = 2$, the numerical approximation to (23) is

(i) Suppose $k = 1$ and $n = 0$

$$\underline{y}_{1,1}(r) = \underline{y}_{1,0}(r) + F_1(1.0, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)),$$

$$y_{1,1}(r) = y_{1,0}(r) + G_1(1.0, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)),$$

such that

$$\underline{y}_{1,1}(r) = c_2(\underline{y}_{1,0}(r)) \tag{24a}$$
$$\overline{y}_{1,1}(r) = c_2(\overline{y}_{1,0}(r)) \tag{24b}$$

where

$$c_2 = \left(1 + \frac{1}{6} + \frac{30}{2^5} + \frac{2 \times 233}{3 \times 2^8} + \frac{103 \times 233}{2^6 \times 2^8} - \frac{63}{3 \times 2^7}\right).$$

We can rewrite (24) as follows:

$$\begin{cases} \underline{y}_{1,1}(r) = c_2 c_1^2(0.75 + 0.25r), \\ \overline{y}_{1,1}(r) = c_2 c_1^2(1.125 - 0.125r). \end{cases}$$

(ii) Suppose $k = 1$ and $n = 1$

$$\underline{y}_{1,2}(r) = \underline{y}_{1,1}(r) + F_1(1.5, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)), \tag{25a}$$
$$y_{1,2}(r) = y_{1,1}(r) + G_1(1.5, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)), \tag{25b}$$

So we have

$$\underline{y}_{1,2}(r) = \underline{y}_{1,0}\left\{1 + \frac{1}{12}(c_2 + 1) + \frac{1}{3}\left(c_2\left(1 + \frac{35}{2^7}\right) + 1\right) + \frac{1}{12}\left(c_2\left(1 + \frac{103}{2^7} + \frac{103 \times 35}{2^7.2^8}\right) + \left(\frac{1}{2^4} + \frac{103}{2^8} - \frac{63}{2^7}\right)\right)\right\},$$

$$\overline{y}_{1,2}(r) = \overline{y}_{1,0}\left\{1 + \frac{1}{12}(c_2 + 1) + \frac{1}{3}\left(c_2\left(1 + \frac{35}{2^7}\right) + 1\right) + \frac{1}{12}\left(c_2\left(1 + \frac{103}{2^7} + \frac{103 \times 35}{2^7.2^8}\right) + \left(\frac{1}{2^4} + \frac{103}{2^8} - \frac{63}{2^7}\right)\right)\right\},$$

Similar to the Case (i), we can rewrite (25) as:

$$\begin{cases} \underline{y}_{1,2}(r) = c_3 c_1^2(0.75 + 0.25r), \\ \overline{y}_{1,2}(r) = c_3 c_1^2(1.125 - 0.125r), \end{cases}$$

where

$$c_3 = 1 + \frac{1}{12}(c_2 + 1) + \frac{1}{3}\left(c_2\left(1 + \frac{35}{2^7}\right) + 1\right) + \frac{1}{12}\left(c_2\left(1 + \frac{103}{2^7} + \frac{103 \times 35}{2^7.2^8}\right) + \left(\frac{1}{2^4} + \frac{103}{2^8} - \frac{63}{2^7}\right)\right).$$

The exact solution of (22) is

$$[x(t)]^r = \begin{cases} [x(1)]^r(3e^{t-1} - 2t), & t \in [1, 1.5],\ 0 \le r \le 1, \\ [x(1)]^r(2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)), & t \in [1.5, 2],\ 0 \le r \le 1. \end{cases}$$

The absolute error between the exact solution and the results obtained by fourth-order RRK, Runge–Kutta method (RK4) Pederson and Sambandham (2008) and Euler method Pederson and Sambandham (2007) is compared in Tables 1 and 2. The approximate solutions and the exact solutions are plotted in Figs. 1 and 2 for the intervals [1, 1.5] and [1.5, 2], respectively. It can be seen that the results of the mentioned method is considerable which prove the efficiency of this method.

Remark 5 As you can see in the Example 1, the number of function evaluations which calculate the approximation solution in $t = 2$ is 16 for the fourth-order Runge–Kutta method,

Table 1 The result of the fourth-order RRK method for Example 1 at $t = 1.5$

r	$\underline{y}^r_{1,1}$	$\underline{y}^r_{1,1}rk4$	$\underline{Eu.}$	$\underline{Y}$	$\lvert\underline{y}^r_{1,1} - \underline{Y}\rvert$	$\lvert\underline{y}^r_{1,1}rk4 - \underline{Y}\rvert$	$\lvert\underline{Eu.} - \underline{Y}\rvert$
0	3.96522	3.96456	3.56288	3.96630	1.08193e−3	1.73498e−3	4.03412e−1
0.1	4.09739	4.09671	3.68165	4.09851	1.11799e−3	1.79281e−3	4.16859e−1
0.2	4.22956	4.22887	3.80041	4.23072	1.15406e−3	1.85064e−3	4.30306e−1
0.3	4.36174	4.36102	3.91917	4.36293	1.19012e−3	1.90848e−3	4.43754e−1
0.4	4.49391	4.49317	4.03793	4.49514	1.22619e−3	1.96631e−3	4.57201e−1
0.5	4.62608	4.62532	4.15670	4.62735	1.26225e−3	2.02414e−3	4.70648e−1
0.6	4.75826	4.75747	4.27546	4.75956	1.29832e−3	2.08197e−3	4.84095e−1
0.7	4.89043	4.88963	4.39422	4.89177	1.33438e−3	2.13981e−3	4.97542e−1
0.8	5.02261	5.02178	4.51299	5.02398	1.37044e−3	2.19764e−3	5.10989e−1
0.9	5.15478	5.15393	4.63175	5.15619	1.40651e−3	2.25547e−3	5.24436e−1
1	5.28695	5.28608	4.75051	5.28840	1.44257e−3	2.31330e−3	5.37883e−1
r	$\overline{y}^r_{1,1}$	$\overline{y}^r_{1,1}rk4$	$\overline{Eu.}$	$\overline{Y}$	$\lvert\overline{y}^r_{1,1} - \overline{Y}\rvert$	$\lvert\overline{y}^r_{1,1}rk4 - \overline{Y}\rvert$	$\lvert\overline{Eu.} - \overline{Y}\rvert$
0	5.94782	5.94684	5.34433	5.94945	1.62290e−3	2.60247e−3	6.05119e−1
0.1	5.88174	5.88077	5.28495	5.88334	1.60486e−3	2.57355e−3	5.98395e−1
0.2	5.81565	5.81469	5.22556	5.81724	1.58683e−3	2.54464e−3	5.91672e−1
0.3	5.74956	5.74862	5.16618	5.75113	1.56880e−3	2.51572e−3	5.84948e−1
0.4	5.68348	5.68254	5.10680	5.68503	1.55077e−3	2.48680e−3	5.78224e−1
0.5	5.61739	5.61646	5.04742	5.61892	1.53273e−3	2.45789e−3	5.71501e−1
0.6	5.55130	5.55039	4.98804	5.55282	1.51470e−3	2.42897e−3	5.64777e−1
0.7	5.48521	5.48431	4.92866	5.48671	1.49667e−3	2.40005e−3	5.58054e−1
0.8	5.41913	5.41823	4.86928	5.42061	1.47864e−3	2.37114e−3	5.51330e−1
0.9	5.35304	5.35216	4.80989	5.35450	1.46061e−3	2.34222e−3	5.44607e−1
1	5.28695	5.28608	4.75051	5.28840	1.44257e−3	2.31330e−3	5.37883e−1

whereas for the fourth-order RRK method, this number is only 12. Also in the Example 2, we will reach to this number as same as in Example 1.

Example 2 Next consider the following hybrid fuzzy IVP,

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), \quad t \in [t_k, t_{k+1}], \\ \;\; t_k = k, \quad k = 0, 1, 2, \ldots, \\ [x(0)]^r = [0.75 + 0.25r,\ 1.125 - 0.125r], \quad 0 \le r \le 1, \end{cases} \tag{26}$$

where

$$m(t) = |\sin(\pi t)|, \quad k = 0, 1, 2, \ldots,$$

$$\lambda_k(\mu) = \begin{cases} \hat{0} & if\ k = 0, \\ \mu & if\ k \in \{1, 2, \ldots\}. \end{cases}$$

Again we apply proposed method in Sect. 4 to approximate the solution of (26) which is as follows:

Case I: when $k = 0$, the solution of (26) in the interval $[0, 1]$:
When $k = 0$, the hybrid fuzzy IVP (26) is:

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_1(x(t_1)), \quad t \in [1, 2], \\ [x(0)]^r = [0.75 + 0.25r, 1.1250.125r], \end{cases} \tag{27}$$

From Theorem 1, we can imply that (27) is equivalent to the ODEs systems as to Example 1. Also the approximate solution of (26) at $t = 1$ is also same value of $y(1.0; r)$ in (21):

$$y(1.0; r) = [(0.75 + 0.25r){c_1}^2, (1.125 - 0.125r){c_1}^2], \\ \quad 0 \le r \le 1,$$

where

$$c_1 = \left(1 + \frac{1}{12} + \frac{41}{96} + \frac{45}{384}\right).$$

Case II: When $k = 1$ the approximate solution of (26) in the interval, $[1, 2]$ is obtained as follows:

The hybrid fuzzy initial value problem (26) is:

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_1(x(t_1)), \quad t \in [1, 2], \\ [x(0)]^r = [0.75 + 0.25r, 1.1250.125r]. \end{cases} \tag{28}$$

Using Theorem 2 and Corollary 3.2 in Pederson and Sambandham (2009), we can conclude that Eq. (28) and systems (23a) and (23b) are equivalent. Now, we apply the fourth-order RRK with $N = 2$ from Sect. 4 to approximate the solution numerically in the interval $t \in [1, 2]$:

Table 2 The result of the fourth-order RRK method for Example 1 at $t = 2$

r	$\underline{y}^r_{1,1}$	$\underline{y}^r_{1,1}rk4$	$\underline{Eu.}$	$\underline{Y}$	$\lvert\underline{y}^r_{1,1} - \underline{Y}\rvert$	$\lvert\underline{y}^r_{1,1}rk4 - \underline{Y}\rvert$	$\lvert\underline{Eu.} - \underline{Y}\rvert$
0	7.25310	7.25182	6.49574	7.25523	2.12484e−3	3.40759e−3	7.59489e−1
0.1	7.49487	7.49355	6.71226	7.49707	2.19567e−3	3.52118e−3	7.84805e−1
0.2	7.73664	7.73528	6.92879	7.73891	2.26650e−3	3.63477e−3	8.10122e−1
0.3	7.97841	7.97700	7.14531	7.98075	2.33732e−3	3.74835e−3	8.35438e−1
0.4	8.22018	8.21873	7.36184	8.22259	2.40815e−3	3.86194e−3	8.60754e−1
0.5	8.46196	8.46046	7.57836	8.46443	2.47898e−3	3.97553e−3	8.86071e−1
0.6	8.70373	8.70219	7.79489	8.70628	2.54981e−3	4.08911e−3	9.11387e−1
0.7	8.94550	8.94391	8.01141	8.94812	2.62064e−3	4.20270e−3	9.36703e−1
0.8	9.18727	9.18564	8.22794	9.18996	2.69146e−3	4.31629e−3	9.62020e−1
0.9	9.42904	9.42737	8.44446	9.43180	2.76229e−3	4.42987e−3	9.87336e−1
1	9.67081	9.66910	8.66099	9.67364	2.83312e−3	4.54346e−3	1.01265e0
r	$\overline{y}^r_{1,1}$	$\overline{y}^r_{1,1}rk4$	$\overline{Eu.}$	$\overline{Y}$	$\lvert\overline{y}^r_{1,1} - \overline{Y}\rvert$	$\lvert\overline{y}^r_{1,1}rk4 - \overline{Y}\rvert$	$\lvert\overline{Eu.} - \overline{Y}\rvert$
0	10.87966	10.87773	9.74361	10.88285	3.18726e−3	5.11139e−3	1.13923e0
0.1	10.75877	10.75687	9.63535	10.76192	3.15185e−3	5.05460e−3	1.12657e0
0.2	10.63789	10.63601	9.52709	10.64100	3.11643e−3	4.99781e−3	1.11391e0
0.3	10.51700	10.51514	9.41882	10.52008	3.08102e−3	4.94101e−3	1.10125e0
0.4	10.39612	10.39428	9.31056	10.39916	3.04560e−3	4.88422e−3	1.08860e0
0.5	10.27523	10.27342	9.20230	10.27824	3.01019e−3	4.82743e−3	1.07594e0
0.6	10.15435	10.15255	9.09404	10.15732	2.97478e−3	4.77063e−3	1.06328e0
0.7	10.03346	10.03169	8.98577	10.03640	2.93936e−3	4.71384e−3	1.05062e0
0.8	9.91258	9.91082	8.87751	9.91548	2.90395e−3	4.65705e−3	1.03796e0
0.9	9.79169	9.78996	8.76925	9.79456	2.86853e−3	4.60025e−3	1.02531e0
1	9.67081	9.66910	8.66099	9.67364	2.83312e−3	4.54346e−3	1.01265e0

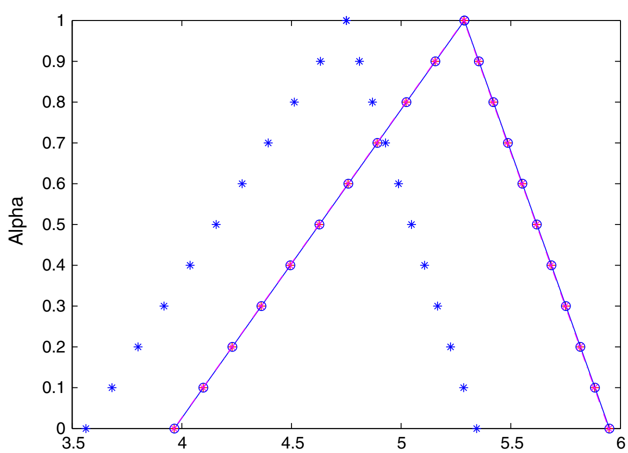

Fig. 1 Exact solution *O*, RRK method *plus*, Runge–Kutta method*cross*, improved Euler method *asterisk*, Example 1 at $t = 1.5$

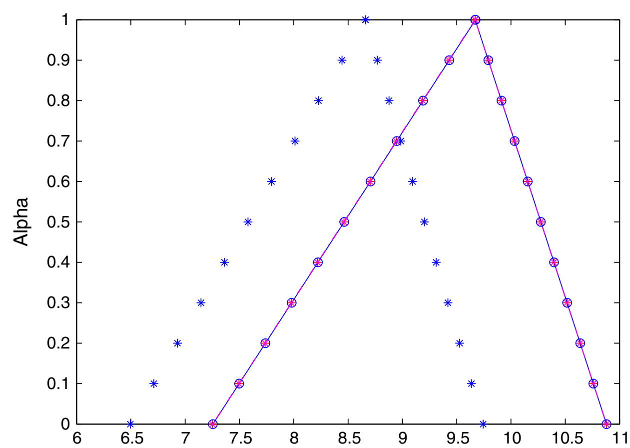

Fig. 2 Exact solution *O*, RRK method *plus*, Runge–Kutta method *cross*, improved Euler method *asterisk*, Example 1 at $t = 2$

(i) Put $k = 1$ and $n = 0$

$$\underline{y}_{1,1}(r) = \underline{y}_{1,0}(r) + F_2(1.0, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)),$$
$$y_{1,1}(r) = y_{1,0}(r) + G_2(1.0, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)),$$

which

$$\underline{y}_{1,1}(r) = c_2(\underline{y}_{1,0}(r)), \tag{29a}$$
$$\overline{y}_{1,1}(r) = c_2(\overline{y}_{1,0}(r)), \tag{29b}$$

in which

Table 3 The result of the fourth-order RRK method for Example 2 at $t = 1.5$

r	$\underline{y}^r_{1,1}$	$\underline{y}^r_{1,1}rk4$	$\underline{Eu.}$	$\underline{Y}$	$\lvert\underline{y}^r_{1,1} - \underline{Y}\rvert$	$\lvert\underline{y}^r_{1,1}rk4 - \underline{Y}\rvert$	$\lvert\underline{Eu.} - \underline{Y}\rvert$
0	4.14664	4.14483	4.02274	4.14377	2.87692e−3	1.06035e−3	1.21022e−1
0.1	4.28486	4.28299	4.15683	4.28189	2.97281e−3	1.09570e−3	1.25056e−1
0.2	4.42309	4.42115	4.29093	4.42002	3.06871e−3	1.13105e−3	1.29090e−1
0.3	4.56131	4.55931	4.42502	4.55814	3.16461e−3	1.16639e−3	1.33124e−1
0.4	4.69953	4.69747	4.55911	4.69627	3.26051e−3	1.20174e−3	1.37159e−1
0.5	4.83775	4.83563	4.69320	4.83439	3.35640e−3	1.23708e−3	1.41193e−1
0.6	4.97597	4.97379	4.82729	4.97252	3.45230e−3	1.27243e−3	1.45227e−1
0.7	5.11419	5.11195	4.96138	5.11065	3.54820e−3	1.30777e−3	1.49261e−1
0.8	5.25242	5.25011	5.09548	5.24877	3.64410e−3	1.34312e−3	1.53295e−1
0.9	5.39064	5.38828	5.22957	5.38690	3.73999e−3	1.37846e−3	1.57329e−1
1	5.52886	5.52644	5.36366	5.52502	3.83589e−3	1.41381e−3	1.61363e−1
r	$\overline{y}^r_{1,1}$	$\overline{y}^r_{1,1}rk4$	$\overline{Eu.}$	$\overline{Y}$	$\lvert\overline{y}^r_{1,1} - \overline{Y}\rvert$	$\lvert\overline{y}^r_{1,1}rk4 - \overline{Y}\rvert$	$\lvert\overline{Eu.} - \overline{Y}\rvert$
0	6.21997	6.12399	6.03412	6.21565	4.31538e−3	9.16646e−2	1.81534e−1
0.1	6.15086	6.06423	5.96707	6.14659	4.26743e−3	8.23568e−2	1.79516e−1
0.2	6.08175	6.00448	5.90003	6.07753	4.21948e−3	7.30489e−2	1.77499e−1
0.3	6.01263	5.94472	5.83298	6.00846	4.17153e−3	6.37411e−2	1.75482e−1
0.4	5.94352	5.88497	5.76593	5.93940	4.12358e−3	5.44332e−2	1.73465e−1
0.5	5.87441	5.82521	5.69889	5.87034	4.07563e−3	4.51254e−3	1.71448e−1
0.6	5.80530	5.76546	5.63184	5.80127	4.02768e−3	3.58175e−2	1.69431e−1
0.7	5.73619	5.70570	5.56480	5.73221	3.97974e−3	2.65097e−2	1.67414e−1
0.8	5.66708	5.64595	5.49775	5.66315	3.93179e−3	1.72018e−2	1.65397e−1
0.9	5.59797	5.58619	5.43071	5.594090	3.88384e−3	7.89403e−3	1.63380e−1
1	5.52886	5.52644	5.36366	5.525028	3.83589e−3	1.41381e−3	1.61363e−1

$$c_2 = \left(1 + \frac{1}{12} + 1.2769 + .2744\right) = 2.6346.$$

(ii) Put $k = 1$ and $n = 1$

$$\underline{y}_{1,2}(r) = \underline{y}_{1,1}(r) + F_2(1.5, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)), \tag{30a}$$

$$\overline{y}_{1,2}(r) = \overline{y}_{1,1}(r) + G_2(1.5, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)). \tag{30b}$$

Then we have

$$\underline{y}_{1,2}(r) = \underline{y}_{1,0}\{(0.6367 + 0.1667 + 0.4245 + 0.1407)c_2 \\ +(0.1344 + 0.0452 - 0.1667)\},$$

$$\overline{y}_{1,2}(r) = \overline{y}_{1,0}\{(0.6367 + 0.1667 + 0.4245 + 0.1407)c_2 \\ +(01344 + 0.0452 - 0.1667)\}.$$

It can be rearranged in the following form:

$$\begin{cases} \underline{y}_{1,2}(r) = c_3 c_1^2(0.75 + 0.25r), \\ \overline{y}_{1,2}(r) = c_3 c_1^2(1.125 - 0.125r), \end{cases}$$

at which

$$c_3 = (0.6367 + 0.1667 + 0.4245 + 0.1407)c_2 \\ +(0.1344 + 0.0452 - 0.1667).$$

For $t \in [1, 2]$, the exact solution of (26) is given by

$$\underline{x}(t; r) = \underline{x}(1; r)\frac{\pi\cos(\pi t)+\sin(\pi t)}{\pi^2+1} + \frac{e^t}{e}x(1; r)\left(1 + \frac{\pi}{\pi^2+1}\right),$$
$$\overline{x}(t; r) = \overline{x}(1; r)\frac{\pi\cos(\pi t)+\sin(\pi t)}{\pi^2+1} + \frac{e^t}{e}x(1; r)\left(1 + \frac{\pi}{\pi^2+1}\right).$$

The results of the fourth-order RRK formula, RK4 and Improved Euler method with $h = 0.5$ at $t = 1.5$ and $t = 2$ are shown in Tables 3 and 4. The exact and approximate solutions by Euler, the new RK4 and RK4 methods are compared and plotted at $t = 1.5$ and $t = 2$ in Figs. 3 and 4. It is deduced that the results of the RRK method are very close to the exact solutions which confirm the validity and feasibility of this method.

Tables 1, 2, 3, 4 reveal that the proposed method is more efficient than the standard Runge–Kutta methods, because of the fact that, the number of function evaluations has been decreased while we reached to the same order of accuracy.

Remark 6 Specifically, the proposed method is more applicable for cases where

- f_y or y'' is cheaper to evaluate than f,
- the use of historical values of f is cheaper then evaluating f, which is satisfied for HFDEs.

Table 4 The result of the fourth-order RRK method for Example 2 at $t = 2$

r	$\underline{y}^r_{1,1}$	$\underline{y}^r_{1,1}rk4$	$\underline{Eu.}$	$\underline{Y}$	$\vert\underline{y}^r_{1,1} - \underline{Y}\vert$	$\vert\underline{y}^r_{1,1}rk4 - \underline{Y}\vert$	$\vert\underline{Eu.} - \underline{Y}\vert$
0	7.73008	7.72924	7.46411	7.73008	8.61512e−3	8.40158e−4	2.65975e−1
0.1	7.98775	7.98689	7.71291	7.98775	8.90229e−3	8.68163e−4	2.74841e−1
0.2	8.24542	8.24453	7.96172	8.24542	9.18946e−3	8.96168e−4	2.83707e−1
0.3	8.50309	8.50217	8.21052	8.50309	9.47663e−3	9.24173e−4	2.92572e−1
0.4	8.76076	8.75981	8.45932	8.76076	9.76380e−3	9.52179e−4	3.01438e−1
0.5	9.01843	9.01745	8.70813	9.01843	1.00509e−2	9.80184e−4	3.10304e−1
0.6	9.27610	9.27509	8.95693	9.27610	1.03381e−2	1.00818e−3	3.19170e−1
0.7	9.53377	9.53274	9.20574	9.53377	1.06253e−2	1.03619e−3	3.28036e−1
0.8	9.79144	9.79038	9.45454	9.79144	1.09124e−2	1.06420e−3	3.36902e−1
0.9	10.04911	10.04802	9.70334	10.04911	1.11996e−2	1.09220e−3	3.45768e−1
1	10.30678	10.30566	9.95215	10.30678	1.14868e−2	1.12021e−3	3.54633e−1
r	$\overline{y}^r_{1,1}$	$\overline{y}^r_{1,1}rk4$	$\overline{Eu.}$	$\overline{Y}$	$\vert\overline{y}^r_{1,1} - \overline{Y}\vert$	$\vert\overline{y}^r_{1,1}rk4 - \overline{Y}\vert$	$\vert\overline{Eu.} - \overline{Y}\vert$
0	11.595133	11.44014	11.19617	11.59513	1.29226e−2	1.54985e−1	3.98963e−1
0.1	11.466298	11.32669	11.07176	11.46629	1.27790e−2	1.39599e−1	3.94530e−1
0.2	11.337464	11.21325	10.94736	11.33746	1.26355e−2	1.24212e−1	3.90097e−1
0.3	11.208629	11.09980	10.82296	11.20862	1.24919e−2	1.08826e−1	3.85664e−1
0.4	11.079794	10.98635	10.69856	11.07979	1.23483e−2	9.34394e−2	3.81231e−1
0.5	10.950959	10.87290	10.57416	10.95095	1.22047e−2	7.80529e−2	3.76798e−1
0.6	10.822124	10.75945	10.44975	10.82212	1.20611e−2	6.26663e−2	3.72365e−1
0.7	10.693289	10.64601	10.32535	10.69328	1.19175e−2	4.72798e−2	3.67932e−1
0.8	10.564455	10.53256	10.20095	10.56445	1.17740e−2	3.18932e−2	3.63499e−1
0.9	10.435620	10.41911	10.07655	10.43562	1.16304e−2	1.65067e−2	3.59066e−1
1	10.306785	10.30566	9.95215	10.30678	1.14868e−2	1.12021e−3	3.54633e−1

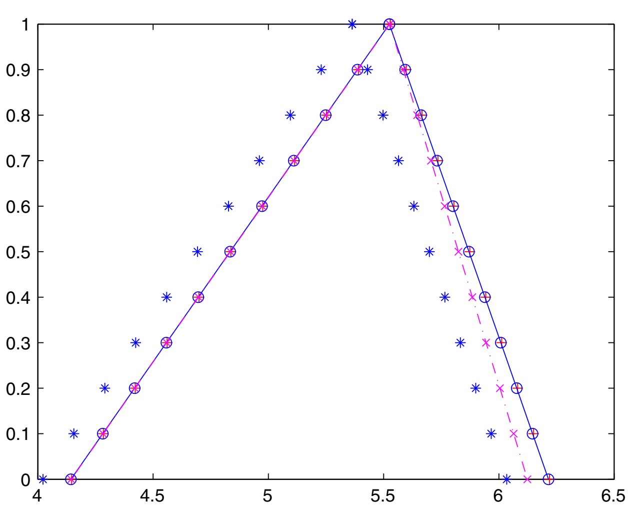

Fig. 3 Exact solution *O*, RRK method *plus*, Runge–Kutta method *cross*, improved Euler method *asterisk*, Example 2 at $t = 1.5$

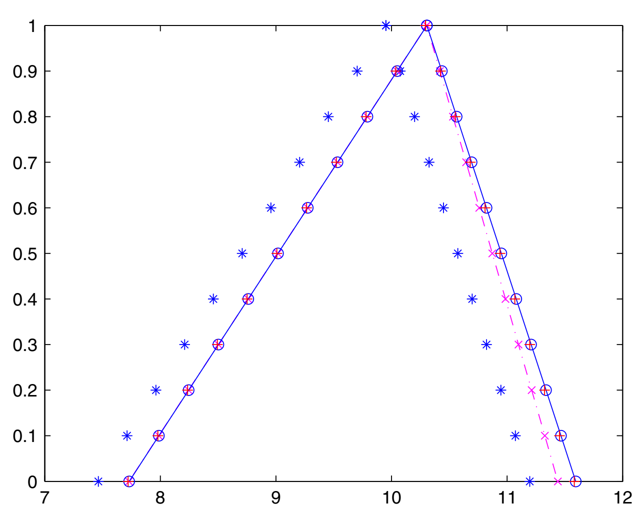

Fig. 4 Exact solution *O*, RRK method *plus*, Runge–Kutta method *cross*, improved Euler method *cross*, Example 2 at $t = 2$

6 Conclusion

In this paper, we apply a family of Runge–Kutta methods which exploit the use of first-order derivatives f'. Specifically, the proposed new formulae with f' ((21), (24) and (25)) are much effective for cases where f' is not computationally expensive to evaluate than f. A clear advantage to this technique is that only three evaluations of f and f' are required per step whereas arbitrary classical Runge–Kutta methods of order three and four used together would require six evaluations of f per step. We achieved not only better accuracy but also same order of convergence and lower functional evaluation in comparison with the classical Runge–Kutta. Results are comparable to Runge–Kutta solution of equal order, thus demonstrating our claim. On the other hand, we solved the HFDEs under Hukuhara differentiability where Bede (2008) proved a characterization theorem that can aid in solving the FDEs numerically. That is by converting the HFDE to a hybrid ODEs system and this will enable us to employ any suitable numerical methods to solve the ODEs.

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