FTFBE: A Numerical Approximation for Fuzzy Time-Fractional Bloch Equation

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***Abstract*—Fractional calculus has a long successful history of 300 years, as it able to model natural phenomena states more accurately than the differential equations of integer order. With this, it plays an important role in variant disciplines. Recently, variant fractional models for the Bloch equations have been proposed, however, effective numerical methods for the fractional Bloch equation (FBE) are still in the infancy stage. In this paper, we extend the time-fractional Bloch equation (TFBE) to fuzzy field under the generalized Caputo differentiability, such that these extensions have natural relationship between crisp. For this purpose, we adopted the fractional Adams-Bashforth-Moulton (FABM) type predictorcorrector method, and introduced a new variant - the fuzzy fractional ADM (FFABM) to find the numerical solution. In this case, a new theorem concerning the error of our proposed FFADM method is also presented. Finally, the capability of the newly developed numerical methods is demonstrated in a fuzzy fractional-order problem, and it achieves satisfactorily in terms of numerical stability.**

***Index Terms*—Fuzzy fractional Bloch equation; Caputo differentiability; Predictor-Corrector method.**

I. Introduction

In contrast to the differential equations of integer order, where the derivatives depend solely on the local behavior of the function, fractional differential equations (FDEs) accumulate the whole information of the function in a weighted form. This is the most significant advantage of fractional order models in comparison with integer order models, in which such effects are neglected. With this, recently, it has lead to numerous applications [1], [21]–[23], [28].

Resultant of this, this topic has gained much interests from mathematicians [12], [13]. It is well known that the exact solutions for most of the FDEs could not be solved easily, thus numerous analytical solutions have been studied extensively to overcome such mathematical complexity. For example, operational matrix method based on the Legendre polynomials [34], He's variational iteration method [32], Adomian's decomposition method [30], [31], fractional Adams method [20] and interpolation functions [25].

On the other hand, modeling of natural phenomena states using mathematical models plays an important role in various disciplines. Commonly the unknown parameters involve in the models are assumed constant over time. In reality, however, some of them are not constant and implicitly depend on several factors. Many of such factors usually do not appear explicitly in the mathematical models due to the tradeoffs between modeling and numerical tractability, and the lack of precise knowledge about them.

In order to deal with such uncertainty in those parameters, stochastic approach is commonly employed with the assumption that stochastic behavior implies knowledge of probabilistic information of the system components. However, this information can be very complicated with errors and vagueness. Alternatively, fuzzy fractional differential equations (FFDEs) has provided another solutions to model the uncertain and/or incompletely specified systems. The first attempt was formulated by Kaleva [24] under the H-differentiability, and subsequently by Bede et al. [15] under the new concept of fuzzy differentiability [14]. According to [15], [16], this new approach seems to be better suited to model the practical situations under uncertainty and imprecision normally present in the real dynamics.

To the best of our knowledge, though various solutions [2], [10], [11], [35], [36] for the FFDEs have been attempted, most of them are based on **exact solutions**; while **numerical methods** for solving the FFDEs are still in the infancy stage, albeit to our previous attempts [3]–[5], [7], [8], [29].

We, therefore, motivate in investigating an effective numerical method with error analysis to approximate the fuzzy time-fractional Bloch equations (FTFBE) on the time interval $J = (0, T]$, with a view to be employed in the image processing domain in near time. The predictor-corrector method was adopted herein due to its simplicity. Particularly, we exploits the fractional Adams-Bashforth as a predictor and the fractional Adams-Moulton as a corrector; and introduced a new variant - the fuzzy fractional Adams-Bashforth-Moulton (FFABM). Finally, we demonstrate the capability of the newly developed numerical methods in a fuzzy fractional-order problem, in terms of accuracy and stability analysis.

The structure of the remainder of this paper is as follows. The related work and motivation is discussed in Section II. In

Section III, some mathematical preliminaries are revisited. In Section IV, an analytical solution of the fuzzy time-fractional Bloch equation (FTFBE) is derived. The proposed fuzzy fractional Adams-Bashforth-Moulton (FFABM) for the FTFBE, as well as the error analysis for the fractional FPCM is detailed in Section V. In Section VI, we present the numerical result that support our theoretical analysis. Finally, we conclude the paper in Section VII.

II. Related Work and Motivation

In 1946, Felix Bloch proposed a set of equations to describe the time dependence of the net magnetization during the course of the nuclear magnetic resonance (NMR) experiment. These equations are known as the *Bloch equations* and give insights into many processes in NMR. The Bloch equations follow first order kinetics and the derivations are first-order differentials which in the classical version take the following form [26], [37]:

$$\frac{d\mathbf{M}(t)}{dt} = \mathbf{M}(t) \times \gamma \mathbf{B}(t) - \mathbf{R}(\mathbf{M}(t) - M_0) \tag{1}$$

in which $\mathbf{M}$ is "bulk" magnetization that arises from all of the magnetic moments in a sample and experiences a torque when placed in a magnetic field and $\mathbf{R}$ is the "relaxation matrix". $\mathbf{B}$ is the magnetic field (in general $\mathbf{B}$ is a vector quantity) and γ is a physical property of each nucleus. For a given abundance, nuclei with higher values of γ produce higher sensitivity NMR spectra.

We note that recently, different fractional models for the Bloch equations have been proposed. For instance, Magin et al. [26] and Velasco et al. [37] have proposed the concept of solutions for Bloch equation with fractional models. They demonstrated that a fractional calculus based diffusion model can be successfully applied to analyze the diffusion images of human brain tissues; as well as new insights of the tissue structures and the micro-environment. Generally, it is difficult to develop robust numerical methods to handle the FBEs in [26], [37], as they are defined based on the fractional operators. Hence, work that similar to us, [39] derive the numerical methods for time-fractional Bloch equations (TFBE) and the anomalous fractional Bloch equations (AFBE), and [19], [27] proposed numerical solutions for FBEs. However, it assumed that the model is in the deterministic state, and effective numerical solutions for the fractional Bloch equation (FBE) are still limited.

According to [15], [16], the applications of the fuzzy concept have appeared more visibly instead of deterministic-stochastic cases. In this paper, we extend the TFBE to fuzzy domain such that these extensions have natural relationship between the crisp and fuzzy cases, as well as a natural relation between fuzzy fractional and fuzzy non-fractional cases. Our assumption is that $\mathbf{M}(t)$ can be seen as a fuzzy function with fuzzy Caputo fractional derivative, ${}_0^cD_t^\alpha$, and uncertain initial conditions. With this, this view is not associated with any of the above hypothesis, although it can include them. It also allows us to consider a wider range of possibilities to incorporate a more diverse behavior and to reflect a non-exactly known parameter. Moreover, in a real-world problem, the unknown quantity $M(t)$ will typically have a certain physical meaning (e.g. a dislocation), however it is not well defined what the physical meaning of a fractional derivative of $M(t)$ is, and hence it is also not clear how such a quantity can be measured. In other words, the essential data merely will not be available in practice. However, the situation is unlike, when we deal with Caputo derivatives. We can specify the initial values $M(0), M'(t), ..., M^{(m-1)}(0)$, i.e., the function value itself and integer-order derivatives. Typically, these data have a well understood physical meaning and could be measured. As we have described above, for the Riemann-Liouville approach this generalization is connected with major practical complications [18]. It is with this motivation that we introduce the fuzzy time-fractional Bloch equations under fuzzy Caputo differentiability.

As a summary, we considered the numerical solution of FTFBE in the literature for the first time. To this end, we introduced the fuzzy fractional Adams-Bashforth (predictor) and the fuzzy fractional Adams-Moulton (corrector) as our proposed numerical scheme with discussions on the error analysis.

III. Preliminaries

We first provide an overview of some common properties of fuzzy settings theory and fuzzy differential equations of integer and fractional order. For detailed, one can refer to [2], [17], [35].

We denote the set of all real numbers by R and the set of all fuzzy number on R is indicated by E. A fuzzy number is a mapping $u : R \to [0, 1]$ with the following properties:

(a) u is upper semi-continuous,

(b) u is fuzzy convex, i.e., for all $x, y \in R, \lambda \in [0, 1]$: $u(\lambda x + (1 - \lambda)y) \geq min\{u(x), u(y)\}$,

(c) u is normal, i.e., $\exists\ x_0 \in R$ for which $u(x_0) = 1$,

(d) supp $u = \{x \in R \mid u(x) > 0\}$ is the support of the u, and its closure cl(supp u) is compact

Definition 3.1: A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:

1) $\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0,
2) $\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0, 1]$, and right continuous at 0,
3) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Reference to Zadeh's extension principle, operation of addition on E is defined as:

$$(u \oplus v)(x) = \sup_{y \in R} \min\{u(y), v(x - y)\}, \quad x \in R, \tag{2}$$

and scalar multiplication of a fuzzy number is given by

$$(k \odot u)(x) = \begin{cases} u(x/k), & k > 0, \\ \widetilde{0}, & k = 0, \end{cases} \tag{3}$$

where $\tilde{0} \in E$. It is well known that the following properties holds for all levels:

$$[u \oplus v]^r = [u]^r + [v]^r \tag{4}$$

$$[k \odot u]^r = k[u]^r \tag{5}$$

Definition 3.2: Let $u \in E$ and $r \in [0,1]$, the r-cut of u is the crisp set $[u]^r$ that contains all elements with membership degree in u greater than or equal to r, i.e.

$$[u]^r = \{x \in \mathbb{R} | u(x) \geq r\} \tag{6}$$

For a fuzzy number u, its r-cuts are closed intervals in $\mathbb{R}$ and can be denoted as

$$[u]^r = [\underline{u}^r, \overline{u}^r] \tag{7}$$

Definition 3.3: The distance $D(u,v)$ between two fuzzy numbers u and v is defined as

$$D(u,v) = \sup_{r \in [0,1]} d_H([u]^r, [v]^r) \tag{8}$$

where $d_H([u]^r,[v]^r) = \max\{|\underline{u}^r - \underline{v}^r|, |\overline{u}^r - \overline{v}^r|\}$ is the Hausdorff distance between $[u]^r$ and $[v]^r$. It is easy to see that d is a metric in E and has the following properties [33]:

(a) $d(u+w, v+w) = d(u,v), \quad \forall u,v,w \in E$,
(b) $d(ku, kv) = |k| d(u,v), \quad \forall k \in R, u,v \in E$,
(c) $d(u+v, w+e) \leq d(u,w) + d(v,e), \quad \forall u,v,w,e \in E$,
(d) (d, E) is a complete metric space

Definition 3.4: Let $x, y \in E$, if there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$. In this paper, the sign "$\ominus$" always stands for H-difference, and also note that $x \ominus y \neq x + (-1)y$.

Definition 3.5: ([14]) Let $f : (a,b) \to E$ and $x_0 \in (a,b)$, f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in E$, such that

(i) for all $h > 0$ sufficiently small, there exist $f(x_0+h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0-h)$ and the limits (in the metric d): $\lim_{h \searrow 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0)$

or

(ii) for all $h > 0$ sufficiently small, there exist $f(x_0) \ominus f(x_0+h)$, $f(x_0-h) \ominus f(x_0)$ and the limits (in the metric d): $\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0+h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0-h) \ominus f(x_0)}{-h} = f'(x_0)$

where h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively. It should be mentioned here the function which is satisfied in Case (i) is called as (1)-differentiable function while Case (ii) is known as (2)-differentiable function.

The principal properties of the derivatives can be found in [14]–[16]. In this paper, we make use of the following theorem:

Theorem 3.1: ([16]) Let $f : R \to E$ be a function and denote $f(x;r) = \left[\underline{f}(x;r), \overline{f}(x;r)\right]$, for each $r \in [0,1]$. Then,

1) If f be a (1)-differentiable function, then $\underline{f}(x;r)$ and $\overline{f}(x;r)$ are differentiable functions and $[f'(x)]^r = \left[\underline{f}'(x;r), \overline{f}'(x;r)\right]$,
2) If f be a (2)-differentiable function, then $\underline{f}(x;r)$ and $\overline{f}(x;r)$ are differentiable functions and $[f'(x)]^r = \left[\overline{f}'(x;r), \underline{f}'(x;r)\right]$.

Definition 3.6: ([35]) Let $f \in C(J,E) \cap L^1(J,E)$, the Riemann-Liouville integral of fuzzy-valued function f is defined as

$$({}^{RL}I^{\alpha}_{0+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad x > a, \quad 0 < \alpha \leq 1. \tag{9}$$

Definition 3.7: ([35]) Let $f \in C(J,E) \cap L^1(J,E)$ and $x_0 \in J$ and $\Phi(x) = \frac{1}{\Gamma(1-\alpha)} \int_{0^+}^x \frac{f(t)}{(x-t)^{\alpha}} dt$, $f(x)$ is fuzzy Caputo fractional differentiable of order $0 < \alpha \leq 1$ at x_0, if there exists an element $({}^c D^{\alpha}_{0+} f)(x_0) \in C(J,E)$ such that for all $0 \leq r \leq 1$, $h > 0$,

(i) $({}^c D^{\alpha}_{0+} f)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0+h) \ominus \Phi(x_0)}{h} = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0-h)}{h}$,

or

(ii) $({}^c D^{\alpha}_{0+} f)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0+h)}{-h} = \lim_{h \to 0^+} \frac{\Phi(x_0-h) \ominus \Phi(x_0)}{-h}$

The fuzzy-valued function f ${}^c[1-\alpha]$-differentiable if it is differentiable as in the Definition 3.7, Case (i), and f is ${}^c[2-\alpha]$-differentiable if it is differentiable as in the Definition 3.7, Case(ii).

Theorem 3.2: ([35]). Let $0 < \alpha \leq 1$ and $f \in C(J,E) \cap L^1(J,E)$, then the fuzzy Caputo fractional derivative exists almost everywhere on J and for all $0 \leq r \leq 1$, we have

$$\begin{aligned}({}^c D^{\alpha}_{0+} f)(x;r) &= \left[\tfrac{1}{\Gamma(1-\alpha)} \int_{0^+}^x \tfrac{\underline{f}'^r(t)dt}{(x-t)^{\alpha}}, \tfrac{1}{\Gamma(1-\alpha)} \int_{0^+}^x \tfrac{\overline{f}'^r(t)dt}{(x-t)^{\alpha}},\right] \\ &= [(I^{1-\alpha}_{0^+} D\underline{f}^r)(x), (I^{1-\alpha}_{0^+} D\overline{f}^r)(x)],\end{aligned} \tag{10}$$

when f is (1)-differentiable, and

$$\begin{aligned}({}^c D^{\alpha}_{0+} f)(x;r) &= \left[\tfrac{1}{\Gamma(1-\alpha)} \int_{0^+}^x \tfrac{f'^r_+(t)dt}{(x-t)^{\alpha}}, \tfrac{1}{\Gamma(1-\alpha)} \int_{0^+}^x \tfrac{f'^r_-(t)dt}{(x-t)^{\alpha}},\right] \\ &= [(I^{1-\alpha}_{0^+} Df^r_+)(x), (I^{1-\alpha}_{0^+} Df^r_-)(x)],\end{aligned} \tag{11}$$

when f is (2)-differentiable.

IV. Analytical Solution of the FTFBE

In this section, we derive the analytical solutions for the FTFBE. With this, we can ascertain the accuracy of our proposed numerical solutions.

The FTFBE can be written as:

$${}^c_0 D^{\alpha}_t X(t) = AX(t) + f(t) \tag{12}$$

with initial condition $X(0) = X_0$, and the matrix A stated by

$$A = \begin{bmatrix} -\frac{1}{s_2} & s_0 & 0 \\ -s_0 & -\frac{1}{s_2} & 0 \\ 0 & 0 & -\frac{1}{s_1} \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

Also, $f(t) = (0, 0, \frac{X_0}{s_1})^T$, $X(t) = (X_x(t), X_y(t), X_z(t))^T$ and $X_0 = (X_x(0), X_y(0), X_z(0))^T$ are fuzzy vectors.

Note that, the coefficients of A are expressed as follow:

$$s_0 = \frac{w_0}{\tau_2^{\alpha-1}}, \ \frac{1}{s_1} = \frac{\tau_1^{\alpha-1}}{T_1}, \frac{1}{s_2} = \frac{\tau_2^{\alpha-1}}{T_2}, \quad \alpha \in (0,1] \tag{13}$$

where τ_1 and τ_2 are fractional time constant.

Theorem 4.1: Consider the following condition:

$$\begin{cases} {}_0^cD_t^\alpha X(t) = AX(t), \\ X(0) = X_0 \in E, \end{cases} \tag{14}$$

where $A \in R^{n\times n}$, has a solution under $(1-\alpha)$-differentiability given by

$$X(t) = \int_0^t e_\alpha^{(t-\xi)A} AX_0 d\xi + X_0 \tag{15}$$

and has a solution $(2-\alpha)$-differentiability given by

$$X(t) = X_0 \ominus (-1)\int_0^t e_\alpha^{(t-\xi)A} AX_0 d\xi. \tag{16}$$

Now, in order to find the general solution of FTFBE, we consider the following Banach space:

$$C_\beta{}^F = \left\{ h(t) \in C^F : ||h||_{c_\beta} = ||(t-a)^\beta h(t)||_c < \infty \right\} \tag{17}$$

where $||h||_c = \max_{t\in[a,b]} d(h(t), \tilde{0})$ and C^F is the set of all continuous fuzzy-valued functions.

Then, using Definition 3.5, Definition 3.7, Theorem 3.2 and Theorem 4.1, we can obtain an explicit general solution of the Eq. 12 (FTFBE) by Theorem 4.2.

Theorem 4.2: Consider the following condition:

$$\begin{cases} {}_0^cD_t^\alpha X(t) = AX(t) + f(t), \\ X(0) = X_0 \in E, \end{cases} \tag{18}$$

where $A \in R^{n\times n}$ and $f \in C_{1-\alpha}[0,T]$, has a solution under $(1-\alpha)$-differentiability given by

$$X(t) = X_0 + \int_0^t e_\alpha^{(t-\xi)A}[f(\xi) + AX_0]d\xi \tag{19}$$

that is valid when A is positive and has a solution $(2-\alpha)$-differentiability given by

$$X(t) = X_0 \ominus (-1)\int_0^t e_\alpha^{(t-\xi)A}[f(\xi) + AX_0]d\xi \tag{20}$$

that is valid when A is negative.

Remark 4.1: In fact we can rewrite the exact solutions under $(1-\alpha)-$ and $(2-\alpha)$-differentiability as follows:

$$X(t) = \int_0^t e_\alpha^{(t-\xi)A} f(\xi)d\xi + [At^\alpha E_{\alpha,\alpha+1}(t^\alpha A) + I]X_0, \tag{21}$$

and

$$X(t) = X_0 \ominus (-1)\int_0^t e_\alpha^{(t-\xi)A} f(\xi)d\xi + [At^\alpha E_{\alpha,\alpha+1}(t^\alpha A)] \tag{22}$$

V. Fuzzy Fractional Predictor-Corrector Method

In this section, we investigate the numerical solutions of the FTFBE using our proposed fuzzy fractional predictor-corrector method (FFPCM). For this purpose, let us consider the FTFBE with initial condition:

$$\begin{cases} {}_0^cD_t^\alpha X(t) = -K_1X(t) + f(t), \\ X(0) = X_0 \in E, \end{cases} \tag{23}$$

where X_0 is a fuzzy initial vector condition and $\alpha \in (0,1]$. It is easy to verify that this problem is equivalent to the following fuzzy Volterra integral equation under $(1-\alpha)$-differentiability

$$X(t) = X_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}[-K_1X(\xi) + f(\xi)]d\xi \tag{24}$$

and under $(2-\alpha)$-differentiability, we have

$$X(t) = X_0 \ominus (-1)\frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}[-K_1X(\xi) + f(\xi)]d\xi \tag{25}$$

In this paper, we adopted the fractional Adams-Bashforth as the predictor and the fractional Adams-Moulton as the corrector formulas [38]; and derive a novel fuzzy fractional Adams-Bashforth (predictor) as well as fuzzy fractional Adams-Moulton (corrector). Here, due to the space constraints, we only consider the state the numerical method under $(2-\alpha)$-differentiability.

We formulate the fuzzy fractional Adams-Bashforth approach, the predictor X_{k+1}^P as follows:

$$X_{k+1}^P = X_0 \ominus (-1)\frac{1}{\Gamma(\alpha)}\sum_{j=0}^k m_{j,k+1}[-K_1X_j + f(t_j)] \tag{26}$$

where

$$m_{j,k+1} = \frac{\tau^\alpha}{\alpha}[(k+1-j)^\alpha - (k-j)^\alpha].$$

Note that, for sake of simplicity we used a uniform discrete scheme $t_j = j\tau,\ j = 0,1,...,n$ and $T = n\tau$, where T is the final time. Also, we formulate the fuzzy fractional Adams-Moulton approach as the corrector by

$$\begin{aligned} X_{k+1} = X_0 \ominus (-1)\tfrac{1}{\Gamma(\alpha)}(\textstyle\sum_{j=0}^k a_{j,k+1}[-K_1X_j + f(t_j)] \\ a_{k+1,k+1}[-K_1X_{k+1}^P + f(t_{k+1})]), \end{aligned} \tag{27}$$

where

$$a_{j,k+1} = \frac{\tau^\alpha}{\alpha(\alpha+1)}\times$$

$$\begin{cases} k^{\alpha-1} - (k-\alpha)(k+1)^\alpha, & j = 0, \\ (k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} & \\ -2(k-j+1)^{\alpha+1}, & 1 \le j \le k, \\ 1, & j = k+1. \end{cases}$$

A. Error analysis of FFPCM

Here, we present a theorem concerning the error of our FFPCM.

Lemma 5.1: Suppose that $f \in C^1[0,T]$, then

$$d(\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} f(t)dt, \sum_{j=0}^{k} m_{j,k+1} f(t_j)) \leq \frac{1}{\alpha} d_\infty(z', \tilde{0}) t_{k+1}^{\alpha} \tau, \tag{28}$$

where $d_\infty(z,\tilde{0}) = \max_{0\leq t\leq T} d(z(t),\tilde{0})$.

Lemma 5.2: Suppose that $f \in C^2[0,T]$, then we have

$$d(\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} f(t)dt, \sum_{j=0}^{k} a_{j,k+1} f(t_j)) \leq N_\alpha d_\infty(z'', \tilde{0}) t_{k+1}^{\alpha} \tau^2, \tag{29}$$

where N_α is a constant depends on α.

Theorem 5.1: Suppose the ${}_0^cD_t^\alpha X \in C^2[0,T]$, then

$$\max_{0\leq j\leq n} d(X(t_j), X_j) = O(\tau^{1+\alpha}) \tag{30}$$

VI. Case study

In order to show the efficiency of the proposed FFPCM, we consider the following FTFBE with initial condition:

$$\begin{cases} {}_0^cD_t^\alpha X(t) = -K_1 X(t) + f(t), \\ X(0) = 0, \end{cases} \tag{31}$$

where $0 < \alpha \leq 1$, $K_1 > 0$ and $f(t) = C(K_1 t^\alpha + \Gamma(1+\alpha))$. The exact solution under $(2-\alpha)$-differentiability is obtained as follows:

$$X(t) = Ct^\alpha \tag{32}$$

where $C = (-1+r, 1-r)$. The exact solution of Eq. 31 is shown for different r-cuts in Fig. 1 with $K_1 = 1$ and $\alpha = 0.5$. Now, in order to obtain the numerical solution in compare of the given exact solution we used the parameter $\alpha = 0.7$, $K_1 = 1$. The comparison between exact and approximate solution of the model is depicted in Fig. 2. Indeed in 0-cut position, the approximation of the upper solution coincide with the deterministic case proposed by Yu et al. [39] which demonstrate the effectiveness of the method in the fuzzy sense. Also, absolute errors between fuzzy approximate solution and the corresponding exact solutions, i.e. $[N_e]^r = [N_{1e}^r, N_{2e}^r] = \left[|\underline{X}_n^r - \underline{X}^r|, |\overline{X}_n^r - \overline{X}^r|\right]$ are shown in Fig. 3. It is clearly seen that the proposed solution is in excellent agreement with the fuzzy exact solution.

VII. Conclusion

In this paper, we considered the numerical solution of fuzzy time-fractional Bloch equation (FTFBE) in the literature for the first time. To this end, an analytical solution and an effective fuzzy predictor-corrector method (FPCM) for solving the FTFBE have been derived. Experiment using a fuzzy fractional-order problem has demonstrated the capability of the newly proposed method. In future, we will consider (i) some other cases for A such that it is neither negative nor positive, (ii) the numerical method under $(1-\alpha)$-differentiability and (iii) application in image processing domain [6], [9].

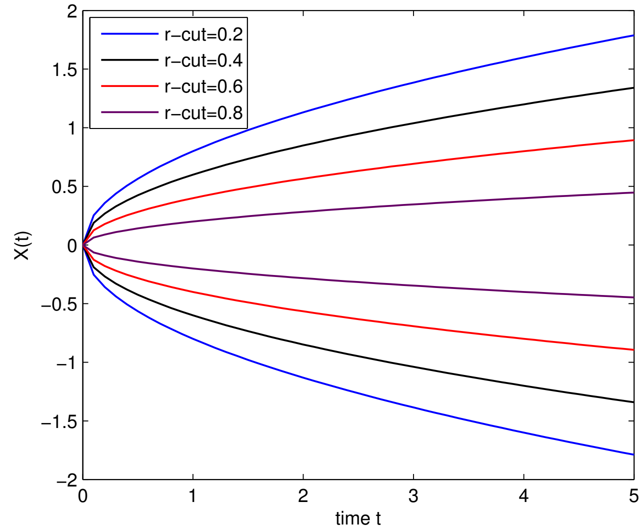

Fig. 1. Exact solution of Eq. 31 for different r-cuts, T=5.

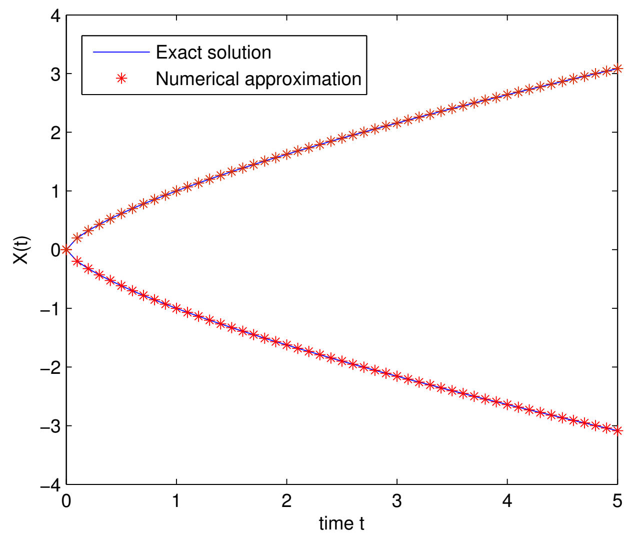

Fig. 2. Comparison of the exact solution of Eq. 31 and the numerical solution using the proposed FFPCM for $\alpha = 0.7$, $K_1 = 1$, $r-cut = 0$.

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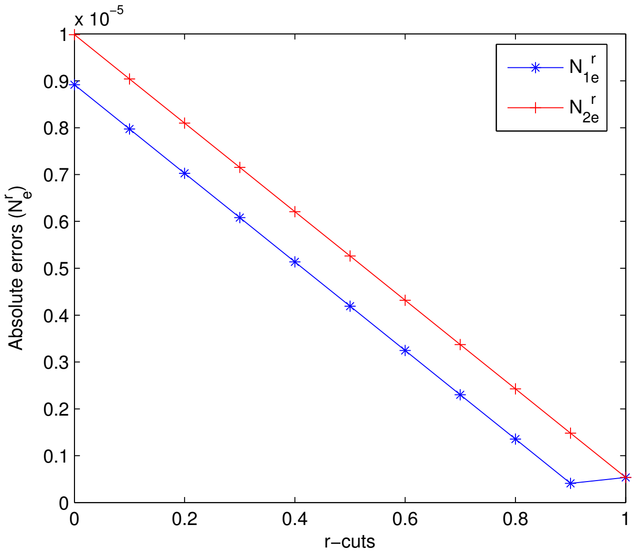

Fig. 3. Absolute errors of the FFPCM for Eq. 31 with $\alpha = 0.7$, $K_1 = 1$, $r - cut = 0$.

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